FY2001 UNIVERSITY SOFTWARE INITIATIVE PROPOSAL
for the
NASA SOFTWARE IV&V FACILITY

Initiative Title: *Integrating Formal Methods and Testing in a Quantitative Software Reliability Assessment Framework*

WHITE PAPER

Inclusion of Risk factors and Failure Severity into the Bayesian Hypothesis Testing Framework for Software Assessment

Bojan Cukic[1], Erdogan Gunel[2], Harshinder Singh[2]

[1]Department of Computer Science and Electrical Engineering
[2]Department of Statistics

West Virginia University
Morgantown, WV 26506-6109
E-mail: cukic@csee.wvu.edu

1. Introduction

This report extends the Bayesian Hypothesis Testing framework for software reliability certification presented in the project's final report of FY2001. The basic statistical framework is described in Section 2. The other sections of this report describe our newly developed features. Section 3 presents testing requirements when observed faults fall into different severity classes. Section 4 introduces the cost functions measuring potential losses if wrong decisions pertinent to software system release are made. Section 5 presents the theory of determining the number of tests under the constraint of minimizing operational risks for the deployed system. The current version of the report is theoretical in nature and presents complex statistical derivations. However, the consequences of these results are poised to radically change the practice of software certification. Practical implications of the proposed methodology will be described in detail in the future reports.

2. Bayesian Hypothesis Testing Framework

Let r denote the number of failures in n tests and θ be the probability of failure. We want to determine n such that $P(\theta \leq \theta_o \mid r, n) = C_o$ where $0 < \theta_0 < 1$ is a given constant.. That is we want to find the number of tests such that when in n tests there is no failure then we are C_o100% confident that the failure probability is less than or equal to some specified threshold value.

We consider testing the null hypothesis

H_o : $\theta \leq \theta_o$

against the alternative hypothesis

H_1 : : $\theta > \theta_o$

for some given constant $0 < \theta_0 < 1$

In Bayesian analysis posterior probability $P(H_o \mid r, n)$ is used to decide between H_0 and H_1. It represents the probability of the null hypothesis in light of the data and prior knowledge. Let $P(H_o)$ and $P(H_1)$, where $P(H_o)+P(H_1)=1$, denote the prior probabilities assigned to null and the alternative hypothesis,

$$O(H_o) = P(H_o) / P(H_1) \tag{1}$$

is called the prior odds of H_o to H_1 and

$$O(H_o \mid r, n) = P(H_o \mid r, n) / P(H_1 \mid r, n) = P(\theta \leq \theta_o \mid r, n) / P(\theta > \theta_o \mid r, n) \tag{2}$$

is called the posterior odds ratio of H_o to H_1. The Bayes factor $F(H_o, H_1)$ is defined as the ratio of posterior odds to prior odds in favor of the null hypothesis,

$$F(H_o, H_1) = O(H_o \mid r, n) / O(H_o) \tag{3}$$

The Bayes factor depends on the prior probability density function $g(\theta)$ of θ and $g(\theta)$ is

$$g(\theta) = \begin{cases} P(H_o)\, g_0(\theta) & \text{if } \theta \leq \theta_o \\ P(H_1)\, g_1(\theta) & \text{if } \theta > \theta_o \end{cases} \tag{4}$$

where go and g_1 are proper probability density functions ($g_o(\theta)>0$, $\int_0^{\theta_0} g_0(\theta)d\theta = 1$, $g_1(\theta)>0$, $\int_{\theta_0}^{1} g_1(\theta)d\theta = 1$) . They describe the distribution of θ over the two hypotheses.

The probability mass function of r given n and θ is the binomial probability $f(r \mid \theta, n) = C_r^n \theta^r (1-\theta)^{n-r}$ if r=0,1,…,n and zero elsewhere, $C_r^n = \dfrac{n!}{r!(n-r)!}$. It can be shown that the Bayes factor is

$$F(H_0, H_1) = \frac{\int_0^{\theta_o} f(r \mid \theta, n) g_o(\theta) d\theta}{\int_{\theta_o}^{1} f(r \mid \theta, n) g_1(\theta) d\theta} \tag{5}$$

In the numerator, f(r | θ, n) is weighted by the prior distribution of θ under the null hypothesis and in the denominator it is weighted by the prior distribution of θ under the alternative hypothesis. If $F(H_o, H_1) > 1$ then the data gives evidence in favor of H_o and if $F(H_o, H_1) < 1$ then we have evidence against H_o .

If we require $P(\theta \le \theta_o \mid r, n) = C_o$, this is equivalent to requiring

$$F(H_o, H_1) = \left(\frac{C_o}{1-C_o}\right) \frac{P(H_1)}{P(H_o)} \ . \tag{6}$$

If θ has a prior distribution on the interval (a, b) where a<b, with the following probability density function

$$f(\theta \mid \alpha, \beta) = \frac{1}{B(\alpha,\beta)(b-a)^{\alpha+\beta-1}} (\theta - a)^{\alpha-1} (b-\theta)^{\beta-1} \tag{7}$$

where α>0, β>0 , B(α, β) = Γ(α) Γ(β) / Γ(α+β) , then it is said to have a Beta distribution on (a,b) with parameters α and β . Symbolically we write $\theta \sim Beta_{(a.b)}(\alpha,\beta)$. The tractable and rich family of probability distributions for θ under H_o and H_1 are $Beta_{(0,\theta_o)}(\alpha_o, \beta_o)$ and $Beta_{(\theta_o,1)}(\alpha_1, \beta_1)$ distributions. If $\alpha_o=\beta_o=1$ then θ has a uniform distribution on $(0, \theta_o)$ under H_o , $g_o(\theta)=1/\theta_o$ if $0<\theta<\theta_o$ and zero elsewhere. Similarly if If $\alpha_1=\beta_1=1$ then θ has a uniform distribution on $(\theta_o, 1)$ under H_1 ,

$g_1(\theta)=1/(1-\theta_o)$ if $\theta_o < \theta < 1$ and zero elsewhere.

Suppose no failures encountered during testing (r=0), we want to determine the number of tests required so that we are $100C_o$% confident that $\theta \le \theta_o$. That is we want to determine n such that $P(\theta \le \theta_o \mid r, n) = C_o$, for some C_o close to one. Suppose we take non-informative uniform prior distributions for θ under H_o and H_1 .

$\theta \mid H_o \sim$ Uniform $(0, \theta_o)$ and $\theta \mid H_1 \sim$ Uniform$(\theta_o,1)$. Bayes factor when r=0 is

$$F(H_o, H_1) = \frac{(1-\theta_o)[1-(1-\theta_o)^{n+1}]}{\theta_o(1-\theta_o)^{n+1}} \tag{8}$$

Since $P(\theta \le \theta_o \mid r, n) = C_o$ if and only if $F(H_o, H_1) = \left(\frac{C_o}{1-C_o}\right)\frac{P(H_1)}{P(H_o)}$ we need to solve

$\frac{(1-\theta_o)[1-(1-\theta_o)^{n+1}]}{\theta_o(1-\theta_o)^{n+1}} = \left(\frac{C_o}{1-C_o}\right)\frac{P(H_1)}{P(H_o)}$ for n. We have

$$n = -\frac{\ln[\frac{C_o\theta_o P(H_1)}{(1-C_o)(1-\theta_o)P(H_o)}+1]}{\ln(1-\theta_o)} - 1 . \tag{9}$$

Table I gives the number of tests for C_o=0.99and various values of θ_o and $P(H_o)$.

Table I. Number of Tests When There is no Failure.

θ_o	$P(H_o)$	n
.01	.01	457
.001	.01	2378
.0001	.01	6831
.00001	.01	9349
.000001	.01	9752
.01	.02	388
.001	.02	1766

.0001	.02	3954
.00001	.02	4736
.000001	.02	4838
.01	.1	228
.001	.1	636
.0001	.1	853
.00001	.1	886
.000001	.1	890
.01	.2	159
.001	.2	333
.0001	.2	387
.00001	.2	394
.000001	.2	395
.01	.3	119
.001	.3	207
.0001	.3	227
.00001	.3	230
.000001	.3	230
.01	.4	90
.001	.4	138
.0001	.4	146
.00001	.4	147
.000001	.4	147
.01	.5	68
.001	.5	93
.0001	.5	98

.00001	.5	98
.000001	.5	98
.01	.6	50
.001	.6	63
.0001	.6	65
.00001	.6	65
.000001	.6	65
.01	.7	34
.001	.7	41
.0001	.7	41
.00001	.7	41
.000001	.7	41
.01	.8	21
.001	.8	23
.0001	.8	24
.00001	.8	24
.000001	.8	24
.01	.9	9
.001	.9	10
.0001	.9	10
.00001	.9	10
.000001	.9	10

Furthermore if $P(H_o)=\theta_o$, $P(H_1)=1-\theta_o$ then

$$n = \frac{\ln(1 - C_0)}{\ln(1 - \theta_o)} - 1 \tag{10}$$

This result in (10) is previously obtained by Bojan ( ), by taking a uniform prior distribution for θ on (0, 1). If g(θ)=1 for 0<θ<1 and zero elsewhere then, this implies that $P(H_o)=\theta_o$, $P(H_1)=1-\theta_o$ and from (4), $g_o(\theta)\, P(H_o)=1$ for $0<\theta<\theta_o$ and , $g_1(\theta)\, P(H_1)=1$ for $\theta_o<\theta<1$ or that θ has uniform distribution on (0, θ_o) under H_o and a uniform distribution on (θ_o, 1) under H_1 . Hence (10) is a special case of (9). Since θ_o is a number close to zero (such as 10^{-3} , 10^{-4} etc) and since taking a uniform distribution on (0,1) for θ implies $P(H_o)=\theta_o$ (very small prior probability is assigned to Ho), it will require a very large n to achieve $P(\theta \leq \theta_o \mid r=0 , n) = C_o$ when C_o is large , C_o=0.95, 0.99 etc.

In our derivation of the number of necessary tests given by (9), taking Uniform(0, θ_o) distribution for θ under H_o makes sense especially when θ_o is very small. However taking uniform (θ_o, 1) distribution for θ under H_1 may not be appropriate. If H_1 was true, we probably feel that expected value of θ under H_1 is close to θ_o and not equal to θo+(1-θ_o)/2 as is the case with Uniform (θ_o, 1) distribution. If $\theta \mid H_1 \sim Beta_{(\theta_o,1)}(\alpha_1, \beta_1)$ then the prior expected value of θ is $E(\theta \mid H_1) = \theta_o + (1-\theta_o)\dfrac{\alpha_1}{\alpha_1+\beta_1}$. If we take α_1=1 and for some small δ take $E(\theta \mid H_1)= \theta_o+ \delta$ then β_1 is $\dfrac{1-\theta_o}{\delta} - 1$ and the necessary number of tests is given by the solution of the following equation for n.

$$\frac{[(n-1)\delta + (1-\theta_o)][1-(1-\theta_o)^{n+1}]}{\theta_o(n+1)(1-\theta_o-\delta)(1-\theta_o)^n} = (\frac{C_o}{1-C_o})\frac{P(H_1)}{P(H_o)} \tag{11}$$

If no restrictions on α_o, β_o, α_1, β_1, are imposed then we need to solve the following equation for n,

$$\frac{B(\alpha_1,\beta_1)}{B(\alpha_o,\beta_o)}(1-\theta_o)\frac{\sum_{i=0}^{n} C_i^n (-\theta_o)^i B(i+\alpha_o+\beta_o)}{B(\alpha_1, n+\beta_1)} = (\frac{C_o}{1-C_o})\frac{P(H_1)}{P(H_o)} \tag{12}$$

If at least one failure is encountered during testing (r >0) and no restrictions on α_o, β_o, α_1, β_1, are imposed then we need to solve the following equation for n,

$$\frac{B(\alpha_1,\beta_1)}{B(\alpha_o,\beta_o)}\frac{\theta_o^r}{(1-\theta_o)^{n-r}}\frac{\sum_{i=0}^{n-r}C_i^{n-r}(-\theta_o)^i B(i+\alpha_o,\beta_o)}{\sum_{i=0}^{r}C_i^r\theta_o^{r-i}(1-\theta_o)^i B(\alpha_1+i,n-r+\beta_1)}=(\frac{C_o}{1-C_o})\frac{P(H_1)}{P(H_o)} \tag{13}$$

. In Table 2. We give the number of tests when no failure encountered, one failure encountered and when two failures are encountered, for some selected values of θ_o and $P(H_o)$. For example, for $\theta_o=10^{-2}$, $P(H_0)=0.4$, if after 90 tests there is no failure then we are 99% confident that $\theta\leq 10^{-2}$. If after 128 tests there is one failure then we are 99% confident that $\theta\leq 10^{-2}$. If after 167 tests there are 2 failures then we are 99% confident that $\theta\leq 10^{-2}$. The Table 2. may also be used in a sequential way: Again suppose that $\theta_o=10^{-2}$, $P(H_0)=0.4$. Perform one failure is encountered then continue testing, if after 128 tests there is one failure then stop, we are 99% confident that $\theta\leq 10^{-2}$. If in 128 tests two failures are observed then continue testing, if after 167 tests there are two failures encountered then stop, we are 99% confident that $\theta\leq 10^{-2}$

Table 2. The number of tests in certification testing.

θ_o	$P(H_o)$	n_o	n_1	n_2
10^{-2}	0.01	457	476	497
10^{-3}	0.01	2378	2671	2975
10^{-4}	0.01	6831	10648	14501
10^{-5}	0.01	9349	33176	63649
10^{-6}	0.01	9752	101273	282007
10^{-2}	0.02	388	410	433
10^{-3}	0.02	1766	2098	2438
10^{-4}	0.02	3954	7549	11315
10^{-5}	0.02	4736	23037	49499
10^{-6}	0.02	4838	70800	221022
10^{-2}	0.1	228	258	289
10^{-3}	0.1	636	1017	1402

10^{-4}	0.1	853	3157	6150
10^{-5}	0.1	886	9646	27281
10^{-6}	0.1	890	30067	123725
10^{-2}	0.4	90	128	167
10^{-3}	0.4	138	411	739
10^{-4}	0.4	146	1251	3260
10^{-5}	0.4	147	3889	14724
10^{-6}	0.4	147	12222	67468
10^{-2}	0.6	50	87	126
10^{-3}	0.6	63	269	552
10^{-4}	0.6	65	827	2458
10^{-5}	0.6	65	2584	11173
10^{-6}	0.6	65	8139	51351

n_0 : Number of tests when no failure encountered
n_1 : Number of tests when one failure is encountered
n_2 : Number of tests when two failures are encountered

3. Determining the number of tests when there are faults with different severity levels.

We will incorpoarate the level of severity of the faults in the determination of the number of tests. Several levels of severity can be identified such as catastrophic , major and minor depending on their impacts to the system service. The definition of severity varies from system to system. Let θ_{Ii} denote the probability of failure due to type i fault, i=1,2,…,k .

We consider testing the null hypothesis

$H_o : \theta_1 \leq \theta_{1o}, \ldots, \theta_k \leq \theta_{ko}$

against the alternative hypothesis

H_1: At least one inequality does not hold

for some given constants $0< \theta_{\iota o}< 1$, i=1,2,…,k . Let r_i denote the number of failures due to type i fault in n tests. We want to find n such that

$P(\theta_1 \le \theta_{1o},\ldots,\theta_k \le \theta_{ko} \mid r_1=0, \ldots, r_k=0, n)=C_o$. That is we want to find the number of tests such that when in n tests there is no failure due to any type of fault then we are C_o 100% confident that each of the failure probabilities of different types of failures are less than or equal to some specified threshold values.

We take the following prior distribution for ($\theta_1,\ldots,\theta_k$),

$$g(\theta_1,\ldots,\theta_k)=\begin{cases} P(H_o)\, g_0(\theta_1,\ldots,\theta_k) & \text{if } (\theta_1,\ldots,\theta_k) \in H_o \\ P(H_1)\, g_1(\theta_1,\ldots,\theta_k) & \text{if } (\theta_1,\ldots,\theta_k) \in H_1 \end{cases} \tag{14}$$

where $g_0(\theta_1,\ldots,\theta_k)$ and $g_1(\theta_1,\ldots,\theta_k)$ are the uniform prior probability density functions under H_o and H_1 ,

$$g_0(\theta_1,\ldots,\theta_k)=\frac{1}{\prod_{i=1}^{k}\theta_{io}} \quad , \quad g_1(\theta_1,\ldots,\theta_k)=\frac{1}{(\frac{1}{k!}-\prod_{i=1}^{k}\theta_{io})} \ . \tag{15}$$

The joint probability mass function of $r_1,\ldots r_k$ given n and $\theta_1,\ldots,\theta_k$ is the multinomial probability,

$$f(r_1,\ldots,r_k \mid \theta_1,\ldots,\theta_k,n)=\frac{1}{r_1!\ldots r_k!(n-r_1-\ldots-r_k)!}\theta_1^{r_1}\ldots\theta_k^{r_k}(1-\theta_1-\ldots-\theta_k)^{n-r_1-\ldots-r_k}$$

It can be shown that the Bayes factor for testing H_o against H_1 is ,

$$F(H_o,H_1)=\frac{(\frac{1}{k!}-\prod_{i=1}^{k}\theta_{io})(1+SUM)}{(\prod_{i=1}^{k}\theta_{io})(\frac{n!\prod_{i=1}^{k}(n+i)}{(n+k)!}-1-SUM)} \tag{16}$$

where,

$$SUM = \sum_{j=1}^{k} (-1)^j \sum{}^{j} (1-\theta_{i_1 o} - ... - \theta_{i_j o})^{n+k} \tag{17}$$

in which the summation $\sum^{j}$ is over all values of ($i_1,\ldots, i_j$) such that i_1=1,…k,….,

i_j=1,…k and $i_1<i_2<\ldots<i_j$. For k=1, $SUM = -(1-\theta_{10})^{n+1}$, for k=2,

$$SUM = -(1-\theta_{10})^{n+2} - (1-\theta_{20})^{n+2} + (1-\theta_{10} - \theta_{20})^{n+2}$$

and for k=3,

$$SUM = -(1-\theta_{1o})^{n+3} - (1-\theta_{20})^{n+3} - (1-\theta_{30})^{n+3}$$
$$+(1-\theta_{10} - \theta_{20})^{n+3} + (1-\theta_{10} - \theta_{30})^{n+3} + (1-\theta_{20} - \theta_{30})^{n+3}$$
$$-(1-\theta_{10} - \theta_{20} - \theta_{30})^{n+3}$$

Note that for k=1, equation in (16) is same as the one in (8).

Requiring $P(\theta_1 \le \theta_{1o},\ldots,\theta_k \le \theta_{ko} \mid r_1=0, \ldots, r_k=0, n)=C_o$ is equivalent to requiring $F(H_o, H_1) = \left(\frac{C_o}{1-C_o}\right)\frac{P(H_1)}{P(H_o)}$. We can find the necessary number of tests by solving this equation for n.

4. The relationship between C_o and the losses under incorrect decisions.

If we take action a when the probability of a fault occurring is θ then the loss that will be incurred is denoted by L(θ, a). Let a_o and a_1 denote the actions of accepting the null hypothesis $H_o : \theta \le \theta_o$ the alternative hypothesis $H_1 : \theta > \theta_o$ respectively. H_o states that the software is reliable where as H_1 states that it is not reliable. Suppose the amount of loss is K_o (in dollars) when we decide that the software as reliable (accept H_o) when it is not reliable(H_1 is true) and the amount of loss is K_1 when we decide that the software is not reliable (accept H_1) when in fact it is reliable (H_o is true). The loss function is

$$L(\theta, a_o) = \begin{cases} 0 & \text{if } \theta \le \theta_o \\ K_o & \text{if } \theta > \theta_o \end{cases}$$

(18)

$$L(\theta, a_1) = \begin{cases} K_1 & \text{if } \theta \le \theta_o \\ 0 & \text{if } \theta > \theta_o \end{cases}$$

We choose the action which has the smaller posterior expectation. Since the posterior expected value of loss for action a_o is $K_o\, P(\theta > \theta_o \mid r, n)$ and the posterior expected value of loss for action a_1 is $K_1\, P(\theta \le \theta_o \mid r, n)$, action ao will be taken (Ho will be accepted) if $K_o\, P(\theta > \theta_o \mid r, n) < K_1\, P(\theta \le \theta_o \mid r, n)$. That is we accept H_o if $P(\theta \le \theta_o \mid r, n) > K_o / (K_o + K_1)$. Since we want $P(\theta \le \theta_o \mid r, n) = Co$, we have $C_o = K_o / (K_o + K_1)$ which shows the relationship between C_o and the losses.

5. Determining the number of tests by minimizing risk

Let r denote the number of failures in n tests and θ be the probability of failure and consider testing the null hypothesis $H_o : \theta \le \theta_o$ against the alternative hypothesis $H_1 : \theta > \theta_o$ for some given constant $0 < \theta_0 < 1$. Let $\delta = \delta(r)$ denote a decision rule. If we observe r failures in n tests then δ is the action that will be taken. Bayes risk of a decision rule with respect to a prior distribution on θ is defined as

$$BR(\delta) = E_\theta[R(\theta,\delta)] = \int R(\theta,\delta) g(\theta) d\theta$$

$$= E_{r|n}[E_{\theta|r,n} L(\theta,\delta,n)] = \sum_{r=0}^{n} \int L(\theta,\delta,n) g(\theta \mid r,n) d\theta h(r) \tag{19}$$

where R(θ, δ) is the risk function

$$R(\theta,\delta) = E_{r|n,\theta}[L(\theta,\delta,n)] = \sum_{r=0}^{n} L(\theta,\delta,n) f(r \mid n,\theta)$$

and

$$f(r \mid \theta, n) = C_r^n \theta^r (1-\theta)^{n-r}$$

$$h(r) = \int f(r \mid n, \theta) g(\theta) d\theta$$

and $L(\theta, \delta, n)$ is the overall loss function. The overall loss is the sum of the loss due to decision and the cost of testing the sofware.

$L(\theta, \delta, n)= L(\theta, \delta) + C(n)$ (20)

where $L(\theta, \delta)$ is the decision loss and $C(n)$ is the cost of testing.

The decision rule which minimizes $BR(\delta)$ is optimal and it is called Bayes decision rule and we will denote it by δ^*. The quantity $BR(\delta^*)$ is called the Bayes risk. Let the actions a_o and a_1 denote accepting H_o and H_1 respectively. The Bayes decision rule is to accept a_o if the posterior expected decision loss of a_o is smaller than that of a_1 , that is if

$$\int_{\theta_o}^{1} L(\theta, a_o) g(\theta \mid r, n) d\theta < \int_{0}^{\theta_o} L(\theta, a_1) g(\theta \mid r, n) d\theta$$

The above inequality implies that we accept H_o if

$$\int_{\theta_o}^{1} L(\theta, a_o) \theta^r (1-\theta)^{n-r} g(\theta) d\theta < \int_{0}^{\theta_o} L(\theta, a_1) \theta^r (1-\theta)^{n-r} g(\theta) d\theta \qquad (21)$$

The above inequality holds if $r<W(n)$ for some constant $W(n)$. We accept H_1 when $r>W(n)$ and $W(n)$ satisfies the following equality,

$$\int_{\theta_o}^{1} L(\theta, a_o) \theta^{W(n)} (1-\theta)^{n-W(n)} g(\theta) d\theta = \int_{0}^{\theta_o} L(\theta, a_1) \theta^{W(n)} (1-\theta)^{n-W(n)} g(\theta) d\theta$$

The Bayes rule δ^* is to accept H_o when $r<W(n)$, accept H_1 when $r>W(n)$ and do not take any action if $r=W(n)$. The decision risk of δ^* is

$$R(\theta, \delta^*) = \begin{cases} L(\theta,a_o)\, P[r<W(n) \mid \theta, n] & \text{if } \theta>\theta_o \\ L(\theta,a_1)\, [\, 1- P[r<W(n) \mid \theta, n] & \text{if } \theta\leq\theta_o \end{cases}$$

The Bayes risk is

$$BR(\delta^*) = \int_{\theta_o}^{1} L(\theta, a_o) P[r < W(n) \mid \theta, n] g(\theta) d\theta$$

$$+ \int_{0}^{\theta_o} L(\theta, a_1)[1 - P[r < W(n) \mid \theta, n] g(\theta) d\theta \tag{22}$$

$$+ C(n)$$

The optimal number of tests is found by minimizing (22) with respect to n. If in particular we take C(n)=n , that is each test costs c, and

$$L(\theta, a_o) = \begin{cases} 0 & \text{if } \theta \leq \theta_o \\ K_o & \text{if } \theta > \theta_o \end{cases}$$

$$L(\theta, a_1) = \begin{cases} K_1 & \text{if } \theta \leq \theta_o \\ 0 & \text{if } \theta > \theta_o \end{cases}$$

$$g(\theta) = \begin{cases} P(H_o) / \theta_o & \text{if } \theta \leq \theta_o \\ P(H_1) / (1-\theta_o) & \text{if } \theta > \theta_o \end{cases}$$

then,

$$BR(\delta^*) = \frac{P(H_1)K_o}{(1-\theta_o)} \sum_{r<W(n)} \int_{\theta_o}^{1} \frac{n!}{r!(n-r)!} \theta^r (1-\theta)^{n-r} \, d\theta$$

$$+ \frac{P(H_o)K_1}{\theta_o} \sum_{r>W(n)} \int_{0}^{\theta_o} \frac{n!}{r!(n-r)!} \theta^r (1-\theta)^{n-r} \, d\theta \tag{23}$$

$$+nc$$

where W(n) satisfies,

$$\frac{P(H_1)K_o}{(1-\theta_o)} \int_{\theta_o}^{1} \theta^{W(n)} (1-\theta)^{n-W(n)} d\theta = \frac{P(H_o)K_1}{\theta_o} \int_{0}^{\theta_o} \theta^{W(n)} (1-\theta)^{n-W(n)} d\theta$$

(24)

For given θ_o, $P(H_o)$, $P(H_1)$, K_o and K_1, we first use the above equality to find W(n) for given n (n=1, 2, 3,…) then compute $BR(\delta^*,n)$ using (23) and examine it to see for what value of n it is minimized.